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## ELECTROPHORETIC THERMAL THEORY

### II. STEADY-STATE RADIAL TEMPERATURE GRADIENTS IN CIRCULAR SECTION COLUMNS

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#### SUMMARY

Central-peripheral temperature gradients are calculated for circular section columns of uniform bore, filled with electrolyte, and carrying current, in the steady state. Wall thickness and material are allowed for. Cases of electrolytes or fillings having zero, positive, and negative temperature coefficients of resistivity are calculated, correction being made for non-uniform current density, using Bessel functions. For negative coefficients, the results differ substantially from simple parabolic theory, gradients being progressively higher than the latter at high heat dissipations.

#### INTRODUCTION

Central-peripheral temperature gradients have been calculated previously, assuming a zero temperature coefficient of resistivity, or uniform density of current through the lumen section. In this paper these gradients are recalculated without these simplifying assumptions, giving substantially different results, in which, for negative coefficients, gradients are progressively larger for higher power dissipations than simple parabolic theory indicates. This is particularly so for increased wall thickness. Similar calculations for rectangular section columns are dealt with in an accompanying paper<sup>1</sup>. A discussion of the anticipated effects of these gradients in electrophoresis, and a comparison of results of digitally computed gradients, in the context of factors determining ideal column shapes and thermal properties is given in a preceding paper<sup>2</sup>. A further paper<sup>3</sup> describes programming and digital computation of the gradients calculated here and in ref. 1. The results of this work were briefly

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reported at a symposium in 1971, by the second author<sup>4</sup>, at whose instigation this project was undertaken.

This treatment of radial temperature gradients was undertaken in order to determine the thermal limitations of columns in transphoretic<sup>5</sup> and isotachophoretic<sup>6,7</sup> displacement<sup>8</sup> electrophoretic methods, both for analytical<sup>9,10</sup> and preparative applications. A second purpose was to lay a basis for calculation of hybrid radial and longitudinal temperature gradients<sup>6</sup> in the steady and unsteady states, and thus of the three-dimensional ionic interface bow in these methods, using current-line and field distortion stabilization theory<sup>11</sup>. These calculations have subsequently been extended to unsteady states by the second author's former colleagues<sup>15</sup>, for the same purposes, under a NASA contract to investigate the displacement method for the preparative separation of cells in zero gravity, the subject of a recent experiment in Skylab<sup>12</sup>. A previous paper deals with the general application of these gradients to electrophoresis<sup>2</sup>.

## THEORY

*For an electrolyte of zero temperature coefficient of resistivity<sup>2</sup>*

Consider a circular section column filled with electrolyte. Assume it to be long enough to neglect end effects in a medial region. Assume heat to be uniformly generated in the electrolyte, and that there is no convective or other fluid movement. Assume a zero temperature coefficient of thermal conductivity and of resistivity. Assume the column exterior to be perfectly cooled and thermostatted, and that the system has reached a steady state. Using the listed symbols, which refer to the column section normal to the long axis,

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dt}{dr} \right) + \frac{W_0}{k_1} = 0$$

for which the general solution is

$$t = A + B \ln r - \frac{W_0 r^2}{4k_1}$$

where  $A$  and  $B$  are integration constants, whence

$$T_2 = A - \frac{W_0 R_2^2}{4k_1} \tag{1}$$

which gives the temperature at any point in the column, the equation being a parabola. For the tube wall,

$$W_0 R_2^2 \ln \left( \frac{R_1}{R_2} \right) = -2 k_2 (T_1 - T_2)$$

So

$$T_2 = T_1 + \frac{W_0 R_2^2}{2 k_2} \ln \left( \frac{R_1}{R_2} \right)$$

and

$$A = T_1 + \frac{W_0 R_2^2}{4 k_1} + \frac{W_0 R_2^2}{2 k_2} \ln \left( \frac{R_1}{R_2} \right)$$

Substituting in eqn. 1 above,

$$t = T_1 + \frac{W_0}{2} \left[ \frac{R_2^2 - r^2}{2 k_1} + \frac{R_2^2}{k_2} \ln \left( \frac{R_1}{R_2} \right) \right] \quad (2)$$

For positive temperature coefficients<sup>2</sup>

$$\frac{d^2 t}{dr^2} + \frac{1}{r} \frac{dt}{dr} = - \frac{W_0}{k_1} (1 + \alpha t) \quad (3)$$

Let

$$\alpha \frac{W_0}{k_1} = \beta^2$$

then

$$\frac{d^2 t}{dr^2} + \frac{1}{r} \frac{dt}{dr} + \beta^2 t = - \frac{W_0}{k_1}$$

the general solution of which is

$$t = A J_0(\beta r) + B Y_0(\beta r) - \frac{W_0}{\beta^2 k_1}$$

But

$$\beta^2 = \alpha \frac{W_0}{k_1}$$

so

$$t = A J_0(\beta r) + B Y_0(\beta r) - \frac{1}{\alpha}$$

where  $r = 0$ ,  $Y_0 = -\infty$  and  $J_0 = 1$ , so

$$t = A J_0(\beta r) - \frac{1}{\alpha} \quad (4)$$

When  $r = R_2$

$$t = T_2 = T_1 + \frac{W_1}{2\pi k_2} \ln \left( \frac{R_1}{R_2} \right)$$

Substituting in eqn. 4 above

$$A J_0(\beta R_2) - \frac{1}{a} = T_1 + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right)$$

therefore

$$A = \frac{1}{J_0(\beta R_2)} \left[ T_1 + \frac{1}{a} + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right) \right]$$

Substituting in eqn. 4 for  $A$

$$t = \frac{J_0(\beta r)}{J_0(\beta R_2)} \left[ T_1 + \frac{1}{a} + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right) \right] - \frac{1}{a} \quad (5)$$

*For negative temperature coefficients<sup>2</sup>*

This is the usual case for electrolytes in aqueous solution. Similarly to eqn. 5 above

$$t = \frac{I_0(\gamma r)}{I_0(\gamma R_2)} \left[ T_1 + \frac{1}{a} + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right) \right] - \frac{1}{a} \quad (6)$$

where

$$\gamma^2 = -a \frac{W_0}{k_1}$$

*The relationship of  $W_t$  and  $W_0$*

Consider an annulus in the lumen section of radius  $r$ , of thickness  $\delta r$  and at a temperature of  $t$  °C.

$$W_t = 2\pi W_0 \int_0^{R_2} (1 + \alpha t) r \, dr$$

Substituting for  $t$ , from eqn. 4 above

$$W_t = 2\pi \alpha W_0 A \int_0^{R_2} J_0(\beta r) r \, dr$$

Let  $\beta r = y$  and  $\beta R_2 = Y$ , then

$$W_t = \frac{2\pi \alpha W_0 A}{\beta^2} \int_0^Y y J_0(y) \, dy$$

But

$$\left[ y \frac{dJ_0(y)}{dy} \right]_0^Y = - \int_0^Y y J_0(y) \, dy$$

Therefore

$$W_t = \frac{-2\pi \alpha W_0 A R_2 J_0'(\beta R_2)}{\beta} \quad (7)$$

When  $r = R_2$

$$t = T_2 = T_1 + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right)$$

Substituting in eqn. 4

$$A = \frac{1}{J_0(\beta R_2)} \left[ T_1 + \frac{1}{\alpha} + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right) \right]$$

Substituting in eqn. 7

$$W_t = \frac{-2\pi \alpha W_0 R_2}{\beta} = \frac{J_0'(\beta R_2)}{J_0(\beta R_2)} \left[ T_1 + \frac{1}{\alpha} + \frac{W_t}{2\pi k_2} \ln\left(\frac{R_1}{R_2}\right) \right]$$

But

$$J_0'(x) = -J_1(x)$$

So for positive  $\alpha$

$$W_t = \left[ \frac{2\pi \alpha W_0 R_2 k_2 J_1(\beta R_2)}{\beta k_2 J_0(\beta R_2) - \alpha W_0 R_2 \ln\left(\frac{R_1}{R_2}\right) J_1(\beta R_2)} \right] \left( T_1 + \frac{1}{\alpha} \right) \quad (8)$$

and for negative  $\alpha$

$$W_t = \left[ \frac{2\pi \alpha W_0 R_2 k_2 I_1(\gamma R_2)}{\gamma k_2 I_0(\gamma R_2) - \alpha W_0 R_2 \ln\left(\frac{R_1}{R_2}\right) I_1(\gamma R_2)} \right] \left( T_1 + \frac{1}{\alpha} \right) \quad (9)$$

#### Current density correction

The foregoing theory is valid only for uniform current density. But current density cannot be uniform for temperature coefficients other than zero. To correct for this, modify eqn. 3

$$\frac{d^2 t}{dr^2} + \frac{1}{r} \frac{dt}{dr} = - \frac{W_0}{k_1} \frac{1}{1 + \alpha t}$$

But this is only numerically soluble. To render this more easily soluble, let

$$\frac{1}{1 + \alpha t} = 1 + \mu t$$

then

$$\frac{d^2 t}{dr^2} + \frac{1}{r} \frac{dt}{dr} = - \frac{W_0}{k_1} (1 + \mu t)$$

the solution of which has been derived for positive and negative values of  $\alpha$  and thus for  $\mu$ , in eqns. 5, 6, 8 and 9. For positive  $\alpha$  and negative  $\mu$

$$t = \frac{I_0(\gamma r)}{I_0(\gamma R_2)} \left[ T_1 + \frac{1}{\mu} + \frac{R_2^2 W_t}{2 k_2} \ln \left( \frac{R_1}{R_2} \right) \right] - \frac{1}{\mu} \quad (10)$$

and

$$W_t = \left[ \frac{2\pi \mu W_0 R_2 k_2 I_1(\gamma R_2)}{\gamma k_2 I_0(\gamma R_2) - \mu W_0 R_2 \ln \left( \frac{R_1}{R_2} \right) I_1(\gamma R_2)} \right] \left( T_1 + \frac{1}{\mu} \right) \quad (11)$$

and for negative  $\alpha$  and positive  $\mu$

$$t = \frac{J_0(\beta r)}{J_0(\beta R_2)} \left[ T_1 + \frac{1}{\mu} + \frac{R_2^2 W_t}{2 k_2} \ln \left( \frac{R_1}{R_2} \right) \right] - \frac{1}{\mu} \quad (12)$$

and

$$W_t = \left[ \frac{2\pi \mu W_0 R_2 k_2 J_1(\beta R_2)}{\beta k_2 J_0(\beta R_2) - \mu W_0 R_2 \ln \left( \frac{R_1}{R_2} \right) J_1(\beta R_2)} \right] \left( T_1 + \frac{1}{\mu} \right) \quad (13)$$

#### Calculation of $\mu$

In practice  $(1 + at)$  is an approximation of a series  $(1 + at + \beta t^2)$ . Plotting the reciprocal of the latter against  $t$ , an approximately straight line is obtained, of gradient  $\mu = 0.033$ , using values for 0.1 mM KCl<sup>14</sup>. This value was therefore used in digital computation<sup>3</sup>. Above 34°, for  $T_1$  of 4°, the relation is less linear and therefore less reliable.

#### CONCLUSIONS

Comparison of equations for zero and negative temperature coefficients of resistivity, where current density changes have to be allowed for in the latter case, shows that for electrolyte solutions the temperature gradient in the steady state is not parabolic. It departs from the parabolic shape increasingly with higher real dissipated power per unit length of column, central temperatures likewise becoming increasingly greater than for zero coefficients. For positive coefficients, the gradients are more favourable. For electrolytes, Joule heating problems are therefore increasingly more severe than simple theory suggests. The general application to electrophoresis is dealt with in a previous paper of the series<sup>2</sup>.

#### SYMBOLS AND UNITS

- $r$  = radius of tube section
- $R_1$  = external radius of tube, cm
- $R_2$  = internal radius of tube, cm
- $k_1$  = thermal conductivity of electrolyte, cal·sec<sup>-1</sup>·cm<sup>-1</sup>·°C<sup>-1</sup>

$k_2$  = thermal conductivity of tube material,  $\text{cal}\cdot\text{sec}^{-1}\cdot\text{cm}^{-1}\cdot^\circ\text{C}^{-1}$

$t$  = temperature

$T_1$  = temperature at wall exterior,  $^\circ\text{C}$

$T_2$  = temperature at lumen periphery,  $^\circ\text{C}$

$T_3$  = temperature at lumen section centre,  $^\circ\text{C}$

$W_0$  = nominal power dissipation at  $T_1$ , assuming a zero value of  $\alpha$ ,  $\text{cal}\cdot\text{cm}^{-3}$

$W_1$  = actual power dissipation per unit length of column,  $\text{cal}\cdot\text{cm}^{-1}$

$J_0$  = Bessel function of zero order and first kind

$J_1$  = Bessel function of first order and first kind

$I_0$  = Bessel function of zero order and modified first kind

$I_1$  = Bessel function of first order and modified first kind

$Y_0$  = Bessel function of zero order and second kind

$\alpha$  = temperature coefficient of resistivity of electrolyte,  $^\circ\text{C}^{-1}$

$$\beta = \sqrt{\frac{\alpha W_0}{k_1}}$$

$$\gamma = \sqrt{-\frac{\alpha W_0}{k_1}}$$

$\mu$  = a function of  $\alpha$

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